MATHEMATICS

PAPER—I

Time Allowed : Three Hours

Maximum Marks : 200

QUESTION PAPER SPECIFIC INSTRUCTIONS

Please read each of the following instructions carefully before attempting questions

There are EIGHT questions in all, out of which FIVE are to be attempted.

Question Nos. **1** and **5** are compulsory. Out of the remaining SIX questions, THREE are to be attempted selecting at least ONE question from each of the two Sections A and B.

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

Answers must be written in ENGLISH only.

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary, and indicate the same clearly.

SECTION—A

1. *(a)* Let $V = \mathbb{R}^4$. Find a basis and dimension of the subspace

$$W = \{(a, b, c, d) \in V : a = b + c,\ c = b + d\}$$ 8

(b) Describe explicitly a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$, which has its range spanned by (1, 0, −1) and (1, 2, 2). 8

(c) Find the relation between the radii of a right circular cylinder and a cone if the former with maximum possible curved surface area is inscribed in the latter. 8

(d) Find the limit of $(\cot x - \tan x)^{\frac{1}{\log_e x}}$, when $x \to 0$. 8

(e) Show that if $ax^2 + 2hxy + by^2 + 2gx + 1 = 0$ represents two straight lines, then $b < 0$ and $bg^2 + h^2 = ab$. 8

2. *(a)* Let $W_1 = \left\{\begin{bmatrix} x & y \\ z & 0 \end{bmatrix} : x, y, z \in \mathbf{C}\right\}$ and $W_2 = \left\{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbf{C}\right\}$ be two subspaces of the vector space of all 2×2 matrices over the complex field **C**. Show that

$$\dim\left(\frac{W_1 + W_2}{W_2}\right) = \dim\left(\frac{W_1}{W_1 \cap W_2}\right)$$ 10

(b) Evaluate the volume of the solid formed by rotating the curve $r = a(1 + \cos\theta)$ about the initial line. 15

(c) *(i)* Reduce the equation $(c^2 + d^2)(x^2 + y^2) = (cx + dy + 2f)^2$ to its canonical form and show that it represents a parabola. Find the latus rectum of the parabola. 6

(ii) A variable sphere passes through the points $(0, 0, \pm c)$ and cuts the lines

$$y - x\tan\theta = 0 = z - c$$

$$y + x\tan\theta = 0 = z + c$$

in the points P and Q. If $|PQ| = 2a$ (where a is a +ve number), then show that the centre of all such spheres lies on the circle $x^2 + y^2 = (a^2 - c^2)\operatorname{cosec}^2 2\theta,\ z = 0$. 9

3. *(a)* If $u = \exp\left\{\sin^{-1}\dfrac{x+y}{\sqrt{x}-\sqrt{y}}\right\}$, then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}u\tan(\log_e u)$. 10

(b) Reduce the matrix

$$A = \begin{bmatrix} 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 8 & 8 & -1 & 26 & 23 \end{bmatrix}$$

to echelon form and then to row canonical form. 15

(c) *(i)* Show that the equation of the plane containing the line $\dfrac{y}{b} + \dfrac{z}{c} = 1$, $x = 0$ and parallel to the line $\dfrac{x}{a} - \dfrac{z}{c} = 1$, $y = 0$ is $\dfrac{x}{a} - \dfrac{y}{b} - \dfrac{z}{c} + 1 = 0$. Further show that if $2d$ is the shortest distance between the given lines, then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{d^2}$$

8

(ii) A variable plane is parallel to the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 0$ and meets the coordinate axes in A, B, C respectively. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0$$

7

4. *(a)* Find the equations of the generating lines of the hyperboloid

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$$

passing through the point $(2, 3, -4)$. 10

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x, y, z) = (5x - y + 3z,\ -6x + 4y - 6z,\ -6x + 2y - 4z)$$

Find all the eigenvalues and corresponding eigenvectors. 15

(c) *(i)* How many loops are generated of the curve $r = a\sin 3\theta$? Find the sum of the areas of all the loops. 8

(ii) Deduce the asymptote of the curve $r\log_e\theta = a$. 7

SECTION—B

5. *(a)* Solve the differential equation

$$p^2+\left(x+y-\frac{2y}{x}\right)p+xy+\frac{y^2}{x^2}-y-\frac{y^2}{x}=0, \text{ where } p=\frac{dy}{dx}$$ 8

(b) Solve the differential equation $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=\cosh x$. 8

(c) A particle moves from rest at a distance a from a centre of force where repulsion at distance x is μx^{-2}. Show that its velocity at distance x is

$$\sqrt{\frac{2\mu(x-a)}{ax}}$$

and that the time it has taken is

$$\sqrt{\frac{a}{2\mu}}\left[\sqrt{x^2-ax}+a\log_e\left(\sqrt{\frac{x}{a}}+\sqrt{\frac{x}{a}-1}\right)\right]$$ 8

(d) Two uniform steel rods of equal size l hang from their junction and rest on a symmetrically placed smooth vertical circular base of radius a. If each of the rods subtends an angle ϕ with the vertical line passing through the centre of the circular base, show that, applying the principle of virtual work, the relation obtained is $l=2a\cot\phi\ \operatorname{cosec}^2\phi$. 8

(e) If $\vec{F}$ is a solenoidal vector, then show that

$$\text{curl curl curl curl } \vec{F}=\nabla^2\nabla^2\vec{F}=\nabla^4\vec{F}$$ 8

6. *(a)* Solve the differential equation

$$\frac{d^2y}{dx^2}+10\frac{dy}{dx}+29y=xe^{5x}+\sin 2x$$ 10

(b) A particle moves in a path so that its acceleration is always directed to a fixed point and is equal to $\frac{\mu}{(\text{distance})^2}$. Show that its path is a conic section and distinguish between the three cases that arise. Further show that the square of the periodic time varies as the cube of the major axis. 15

(c) *(i)* If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then find $\text{curl}\left(\frac{\vec{r}}{|\vec{r}|}\right)$. 6

(ii) Find the curvature and torsion of the curve $x=a\cos t,\ y=a\sin t,\ z=bt$. 9

7. *(a)* If near the surface of a celestial body having atmosphere, the gravity is almost constant and the absolute temperature in its atmosphere is given by

$$T = T_0 \sqrt{1 - \frac{z^2}{n^2 H^2}}$$

H being the height of the homogeneous atmosphere and n a constant quantity, show that the pressure in the atmosphere will be given by

$$p = p_0 \exp\left(\sin^{-1} \frac{z}{nH} \right)$$

where p_0 is the pressure at $z = 0$. 10

(b) Solve the differential equation

$$\frac{d^2 y}{dx^2} - \cot x \frac{dy}{dx} - (1 - \cot x) y = e^x \sin x$$ 15

(c) *(i)* If ϕ satisfies $\nabla^2 \phi = 0$, then show that $\nabla \phi$ is both solenoidal and irrotational. 4

(ii) Verify the divergence theorem for the function

$$\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$$

taken over the parallelepiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. 11

8. *(a)* Verify Green's theorem for

$$\int_C [(xy + y^2)\, dx + x^2 dy]$$

where C is bounded by the curves $y = x$ and $y = x^2$. 10

(b) Using the method of variation of parameters, solve the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = x^2 \log x, \; x > 0$$ 15

(c) Establish a stability criterion if a rigid body is lying on another rigid body at a point of contact, and also both have rough surfaces to prevent sliding and a small area around the point of contact of both of them is circular.

A solid frustum of a paraboloid of revolution of height h and latus rectum $2a$ rests with its vertex on that of a paraboloid of revolution of latus rectum $2b$. Find the stability condition. 15

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